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GRAVITY ASSIST OPTIMIZATION TECHNIQUE APPLICABLE TO A VARIETY OF SPACE MISSIONS

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ABSTRACT

Six different free fall missions that utilize the gravitational influence of a passing planet are presented. The missions are: minimum flight time for deep space probes, maximum inclination out of the ecliptic, maximum component of the velocity normal to the ecliptic, maximum distance out of the ecliptic within a specified time, minimum closest approach to the sun, and attaining a specific value of perihelion. Any mission objective that can be defined as a function of the heliocentric orbital parameters may also be treated. An iterative procedure is used to optimize the post encounter orbit for the desired mission.

The patched conic assumption is used in that the actual flight path of a free fall vehicle is approximated by conic trajectories in finding the post encounter trajectory.

Examples are included of missions utilizing the gravitational influence of the planet Jupiter. Without using Jupiter for an assist such missions would be difficult to perform with our present technology.

TABLE OF CONTENTS

LIST OF SYMBOLS

Notation

τ	region of sphere of influence of target
$\vec{V}_1$	asymptotic approach velocity vector of probe with respect to the sun as it enters τ
m	mass of target planet
μ_P	gravitational constant of planet = Gm
M	mass of sun
μ_S	gravitational constant of sun = GM
$\lvert\vec{B}\rvert$ = BMAG	miss distance
$\vec{V}_1'$	asymptotic approach velocity vector of probe with respect to target as it enters τ
$\vec{V}_2'$	asymptotic departing velocity vector of probe with respect to target as it leaves τ
$\vec{V}_2$	asymptotic departing velocity vector of probe with respect to sun as it leaves τ
$\vec{V}_1^P$	heliocentric velocity of planet when probe enters τ
$\vec{V}_2^P$	heliocentric velocity of planet when probe leaves τ
S	radius of sphere of influence
$\vec{R}_1^P$	heliocentric position of planet when probe enters τ
$\vec{R}_2^P$	heliocentric position of planet when probe leaves τ
r_P	pericenter
r_a	apocenter

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I. INTRODUCTION

When a space vehicle is launched on a free fall trajectory in such a manner that it comes within the vicinity of a planet, the gravitational field of the planet can considerably alter the vehicle's orbit about the sun. The approach trajectory can be established in such a manner that the post encounter trajectory will allow the probe to complete many missions that otherwise could not be accomplished by a direct flight from Earth without extreme energy or payload penalties. For example, if it is desired to fly a solar probe mission, a direct flight must have sufficient energy not only to escape the Earth's gravitational field but also to negate the Earth's orbital velocity. If the probe is launched with a hyperbolic excess velocity of 11 km/sec, the closest it will come to the sun is .25 A.U.[1] If, however, the gravitational influence of the planet Jupiter is used, it is then possible to impact the sun with approximately the same hyperbolic excess velocity.

In a similar manner, many interesting missions may be accomplished which otherwise would be extremely difficult or expensive. It is the purpose of this report to develop a theoretical method of utilizing the gravitational influence of a planet in order to accomplish some of these missions.

II. FLYBY-DETERMINATION OF POST ENCOUNTER CONDITIONS

In determining the post encounter trajectory after it has passed through the gravitational field of a planet, it is assumed that certain pre-encounter conditions are known.

Assuming a fixed transfer time from a park orbit about the earth to the planet or target we may obtain an approximate transfer trajectory to the target as in Reference 2*. This trajectory assumes both the earth and the target planet to be massless points. We obtain as a result the velocity vector ($\vec{V}_1$) of the probe with respect to the sun at encounter with the target.

*Subroutine PLANET

We then take into consideration the gravitational field of the planet and utilize the patched conic assumption which states that at any given time the probe is under the influence of only one body. If m and M are the masses of the planet and sun respectively then the radius (S) of the sphere of influence is given by

$$S = \left(\frac{m}{M}\right)^{2/5} R^P$$

where R^P is the planet's distance from the sun. We have $\vec{V}_1$ and knowing the injection date of the probe and its transfer time we can find $\vec{V}_1^P$ the heliocentric velocity of the target. Reference to Figure 1 shows that $\vec{V}_1' = \vec{V}_1 - \vec{V}_1^P$ where $\vec{V}_1'$ is the velocity of the probe with respect to the planet.

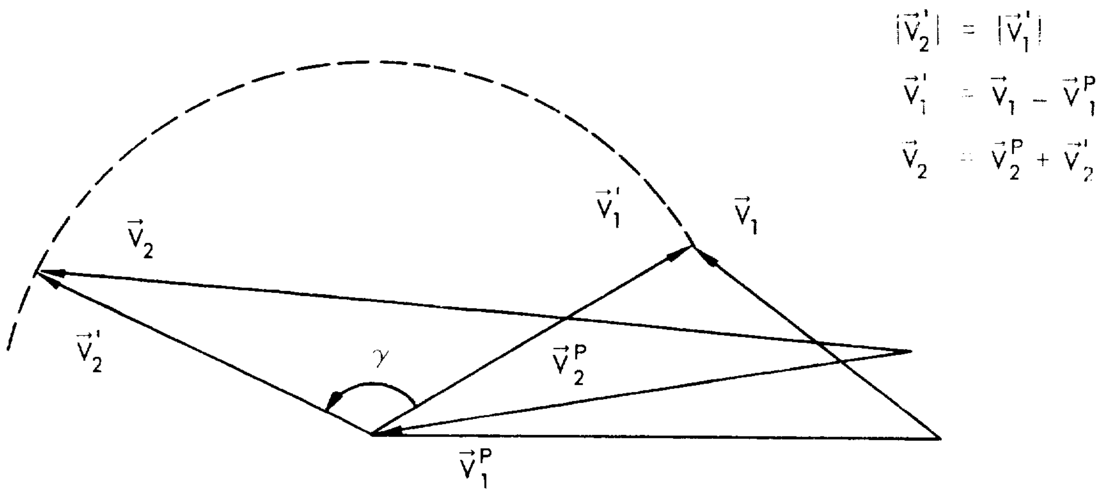


Figure 1

Within the sphere of influence τ, we assume that the probe's trajectory will be hyperbolic with respect to the planet (see Figure 2). We now make a basic and very critical assumption. We assume that $\vec{V}_1'$ is now the velocity of the probe when it enters τ and that we may choose the radius of closest approach of the probe without significantly altering the direction or magnitude of $\vec{V}_1'$. This is equivalent to assuming that while the direction and magnitude of $\vec{V}_1$ are fixed at the calculated value, we are still free to choose the point at which the vehicle pierces the surface of τ (point A in Figure 2). This equivalence is illustrated in Figure 2: for a given asymtote and energy at entrance to τ, the entire trajectory within τ, including the RCA, is determined by the location of point A.

That our assumption is valid is indicated by Table 1 where $\vec{V}_1' = \vec{V}_1 - \vec{V}_1^P$ lies along the $\vec{S}^\circ$ vector (Figure 2). This table was compiled for both retrograde and posigrade flybys of the planet Jupiter and for a radius of closest approach varying from 100,000 to 10,000,000 km.* The values for RCA span a far larger region than we are interested in.

Table 1

Posigrade RCA $\times 10^5$ km	$\vec{S}^\circ$ Right Ascension (DEG)	$\vec{S}^\circ$ Declination (DEG)	$\lvert\vec{V}_1\rvert$ (km/sec)
1	161.38295	-.037895218	13.863505
5	161.36287	-.037826118	13.873625
10	161.34038	-.037764311	13.882193
50	161.10998	-.037458739	13.910651
100	160.75975	-.037179910	13.922712
Retrograde			
1	161.43148	-.037922481	13.863560
5	161.48549	-.037896001	13.873828
10	161.53529	-.037877017	13.882264
50	161.82623	-.037871506	13.910774
100	162.11483	-.037958746	13.923062

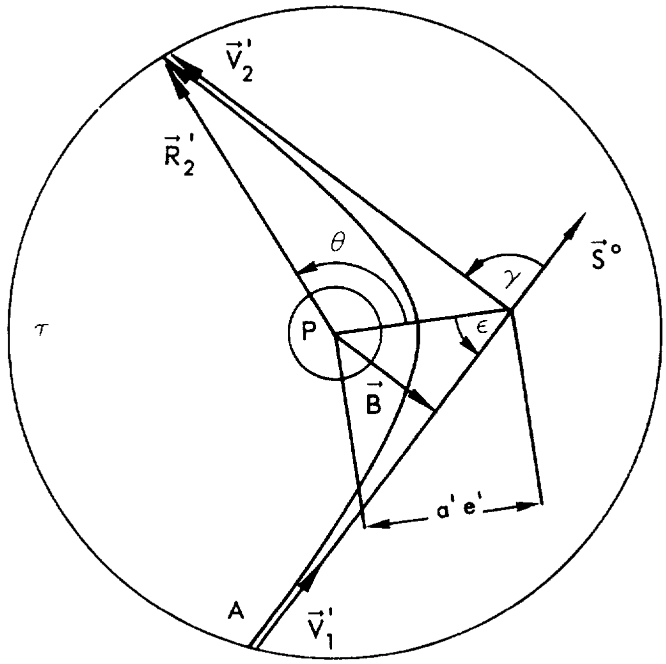
A convenient method for specifying the location of point A and the radius of closest approach is to select the magnitude of the miss vector ($\lvert\vec{B}\rvert$ = BMAG in Figure 2) and an angle ψ, measured between the B vector and a fixed vector in the planet's orbital plane.

Now, the total effect of the encounter is simply to rotate $\vec{V}_1'$ thru an angle γ, as illustrated in Figures 1 and 2, since we must have

$$\lvert\vec{V}_1'\rvert = \lvert\vec{V}_2'\rvert$$

if energy is to be conserved.

*Table 1 was generated using the refine option of the Quick Look Program (Reference 2).



$|\vec{B}| = \text{BMAG}$

$|\vec{R}_2'| = S$

Figure 2

We may readily calculate γ if BMAG is specified:

let a' = semi major axis of the hyperbolic trajectory in τ.

then

$$a' = \frac{\mu_p S}{2\mu_p - S\,|\vec{V}_1'|^2}$$

from the energy equation.

From Figure 2 and the properties of an hyperbola we have the eccentricity

$e' = \dfrac{1}{\cos\epsilon}$ where ϵ is the half angle between the asymptotes

and $\sin\epsilon = \dfrac{|\vec{B}|}{a'\,e'} = \dfrac{\text{BMAG}}{a'}(\cos\epsilon)$

Therefore $\tan\epsilon = \dfrac{\text{BMAG}}{a'}$

Now $\gamma = \pi - 2\epsilon$ or

$$\gamma = \pi - 2\tan^{-1}\left(\frac{\text{BMAG}}{a'}\right)$$

In order to compute $\vec{V}_2'$ we now need to define the second variable ψ which with BMAG will define the entrance point on τ. We establish an orthogonal coordinate system consisting of the unit vectors $\vec{S}^\circ$, $\vec{T}^\circ$, and $\vec{R}^\circ$ in the following manner:

$\vec{S}^\circ = \dfrac{\vec{V}_1'}{|\vec{V}_1'|}$ lies along the incoming asymptote. Let $\vec{k}^\circ$ be a unit vector perpendicular to the planet's orbital plane, then

$\vec{T}^\circ = \vec{k}^\circ \times \vec{S}^\circ$, $\vec{T}^\circ$ lies in the plane of the planet's orbit

$\vec{R}^\circ = \vec{S}^\circ \times \vec{T}^\circ$

Figure 3 shows $\vec{V}_1'$, $\vec{V}_2'$, and $\vec{B}$ in relation to this coordinate system.

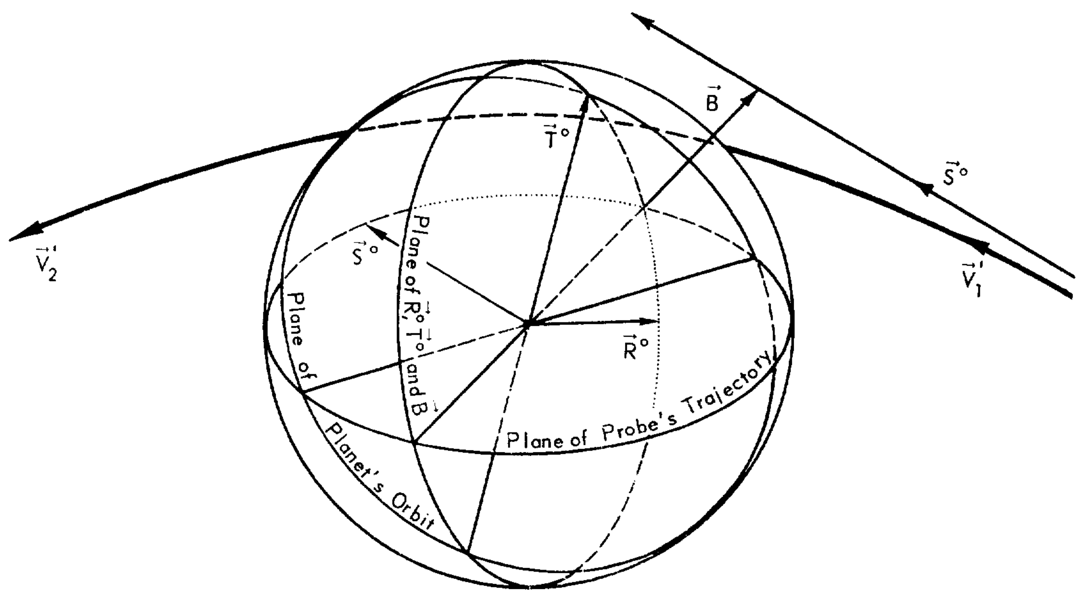


Figure 3

Since $\vec{B} \perp \vec{S}^\circ$, $\vec{B}$ lies in the plane formed by $\vec{T}^\circ$ and $\vec{R}^\circ$. Also, since the hyperbolic orbit lies in a plane $\vec{B}$, $\vec{V}_1'$, and $\vec{V}_2'$ must lie in the same plane. We define our second variable ψ to be the angle between $\vec{T}^\circ$ and $\vec{B}$ measured from $\vec{T}^\circ$ to $\vec{R}^\circ$. Now with reference to Figure 4 we may decompose $\vec{V}_2'$ into its components

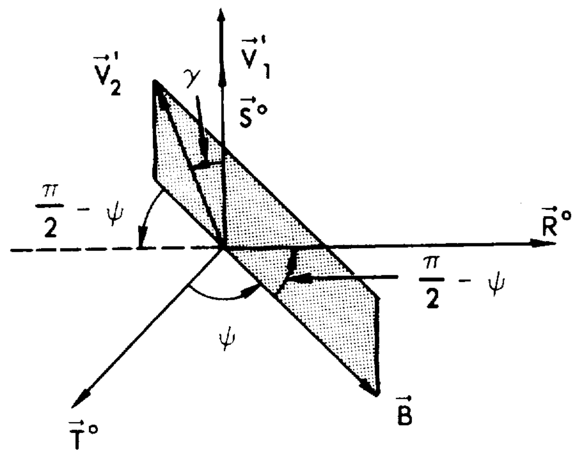


Figure 4

along $\vec{S}^\circ$, $\vec{T}^\circ$, $\vec{R}^\circ$ and obtain

$$\vec{V}_2' = |\vec{V}_1'|\left\{\vec{S}^\circ \cos\gamma - \sin\gamma\left[\vec{R}^\circ \sin\psi + \vec{T}^\circ \cos\psi\right]\right\}.$$

For a first approximation we assume the time spent in τ to be negligible and therefore we let $\vec{V}_2^P = \vec{V}_1^P$, then $\vec{V}_2 = \vec{V}_2' + \vec{V}_1^P$ (the heliocentric departure velocity). Similarly if $\vec{R}_2$ is the position vector of the departing probe we may assume $\vec{R}_2 = \vec{R}_1^P$ where $\vec{R}_1^P$ is the heliocentric position of the planet corresponding to $\vec{V}_1^P$.

We now use the approximate values $\vec{R}_2$ and $\vec{V}_2$ of the probe to obtain a more accurate estimate of the post encounter position and velocity. As before we have

$$\text{Cos } \epsilon = \frac{1}{e'}$$

and $\text{Cos } \epsilon = \sqrt{\frac{1 + \text{Cos } 2\epsilon}{2}} = \frac{1}{e'}$

and with reference to Figure 2 we see that

$$\vec{V}_1' \cdot \vec{V}_2' = |\vec{V}_1'|\ |\vec{V}_2'|\ \text{Cos }(\pi - 2\epsilon) = -\ |\vec{V}_1'|\ |\vec{V}_2'|\ \text{Cos } 2\epsilon$$

then $e' = \sqrt{\frac{2}{1 + \text{Cos } 2\epsilon}}$

$$e' = \sqrt{\frac{2}{1 - \frac{\vec{V}_1^{\,'} \cdot \vec{V}_2^{\,'}}{|\vec{V}_1^{\,'}|\ |\vec{V}_2^{\,'}|}}}$$

$$e' = \sqrt{\frac{2\,|\vec{V}_1^{\,'}|\ |\vec{V}_2^{\,'}|}{|\vec{V}_1^{\,'}|\ |\vec{V}_2^{\,'}| - \vec{V}_1^{\,'} \cdot \vec{V}_2^{\,'}}}$$

$\rho = a' \left(1 - e'^{\,2}\right) =$ the semi latus rectum. To find the true anomaly at exit from τ we use the radius equation and obtain

$$\theta = \mathrm{Cos}^{-1}\left[\frac{\rho - S}{S e'}\right]$$

The time from periapsis to true anomaly θ is calculated,* which is one half the total time spent in τ. With this time, we may step forward and backwards along the heliocentric orbit to obtain new values for $\vec{V}_1$, $\vec{V}_1^{\,P}$, $\vec{V}_2^{\,P}$, $\vec{R}_1^{\,P}$, and $\vec{R}_2^{\,P}$: We then recalculate a', γ, $\vec{S}^\circ$, $\vec{T}^\circ$, $\vec{R}^\circ$, $\vec{V}_2^{\,'}$, ρ and θ exactly as we did before. We repeat this iterative procedure until time from periapsis is within a predetermined tolerance. Then $\vec{V}_2 = \vec{V}_2^{\,'} + \vec{V}_2^{\,P}$ using the values from the last iteration only.

We now need $\vec{R}_2$, the heliocentric position of the probe when it leaves τ. To calculate this we first calculate a unit vector $\vec{B}^\circ$ along $\vec{B}$. From Figure 4

$$\vec{B}^\circ = \mathrm{Cos}\,\psi\,\vec{T}^\circ + \sin\psi\,\vec{R}^\circ$$

and from Figure 2 we find

$$\vec{R}_2^{\,'} = S\left[\vec{B}^\circ\,\mathrm{Cos}\,(\theta + \pi/2 - \epsilon) + \vec{S}^\circ\,\mathrm{Cos}\,(\theta - \epsilon)\right]$$

then $\vec{R}_2 = \vec{R}_2^{\,P} + \vec{R}_2^{\,'}$

From $\vec{R}_2$ and $\vec{V}_2$ we may now calculate the post encounter heliocentric trajectory. From the area integral

$$\vec{R}_2 \times \vec{V}_2 = \vec{H}_2$$

*Reference 2, Subroutine TCONIC.

The Hamilton Integral for the post encounter trajectory is $\vec{V}_2 = \frac{1}{\rho} \vec{H}_2 \times \left(\vec{R}_2^\circ + \vec{e} \right)$ where $\vec{e}$ is a vector in the direction of perigee.

Upon cross multiplying on the right by $\vec{H}_2$ we obtain

$$\vec{e} = \frac{1}{\mu_S} \vec{V}_2 \times \vec{H}_2 - \vec{R}_2^\circ$$

then $e = |\vec{e}|$

also $\rho = \dfrac{\vec{H}_2 \cdot \vec{H}_2}{\mu_S} = a\left(1 - e^2\right)$

and from this we obtain a. The true anomaly is easily obtained from the radius equation $r = \dfrac{a\left(1 - e^2\right)}{1 + e \cos \theta}$ and the inclination by taking the dot product of $\vec{H}_2$ and a unit vector perpendicular to the ecliptic. We now have the standard elements needed to compute the total trajectory.

III. POST ENCOUNTER

Maximum distance out of the ecliptic within a given time

Appendix A describes a procedure for obtaining the maximum distance out of the ecliptic that any given post encounter trajectory attains. When the period of the orbit is large, it may not be realistic for a mission to be based on reaching such a distance. A more useful mission might be to attain the maximum distance possible within a specified time (Δt) from post encounter.

From Δt and the position and velocity vectors ($\vec{R}$ and $\vec{V}$) at departure from τ, the position and velocity $\left(\vec{R}_t, \vec{V}_t\right)$ of the probe at time Δt can be computed.* Figure 5 shows the orbit with respect to the line of intersection with the ecliptic plane where the points P and P' represent the maximum distances out of the ecliptic as discussed in Appendix A.

By defining $\vec{u}^\circ$ to be a unit vector along the line of intersection it is easily seen that

*Reference 2, Subroutine STEPD.

$$D = \left[|\vec{R}|^2 - (\vec{R} \cdot \vec{u}^\circ)^2 \right]^{\frac{1}{2}}$$

$$D_1 = \left[|\vec{R}_t|^2 - (\vec{R}_t \cdot \vec{u}^\circ)^2 \right]^{\frac{1}{2}}$$

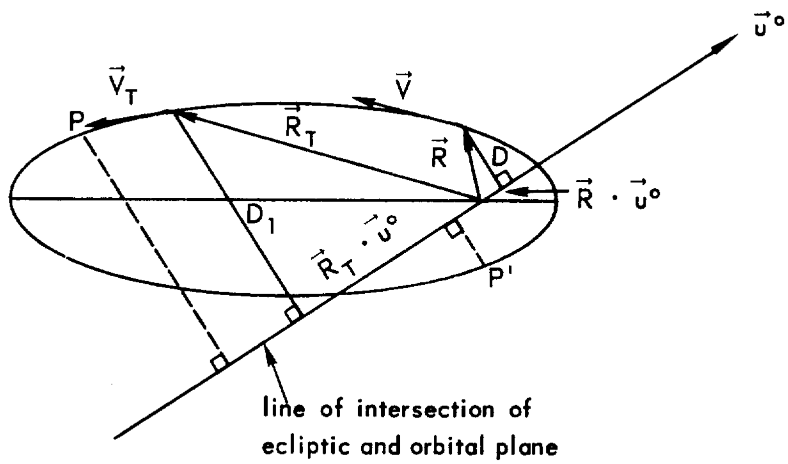


Figure 5

If i is the inclination of the orbit to the ecliptic then the distances out of the ecliptic are

$$d = D \sin i$$

$$d_1 = D_1 \sin i$$

we now have computed four distances out of the ecliptic: the two maxima associated with P and P', and d and d_1. To determine which of these distances is the maximum reached within Δt we define $\vec{k}^\circ$ to be a unit vector perpendicular to the ecliptic and form the dot products $\vec{V} \cdot \vec{k}^\circ$ and $\vec{V}_t \cdot \vec{k}^\circ$. If these quantities are both positive or negative then the maximum distance is the larger of d, d_1. If the dot products are of opposite sign then the proper solution is one of the distances associated with P and P'. The sign of $\vec{V} \cdot \vec{k}^\circ$ determines the correct one. Finally if either dot product is zero then we are already at the maximum.

The distance thus found is the functional value to be used in the optimization process.

Minimum Time for Deep Space Probes

In this option we compute the time from post encounter to a given heliocentric distance (x A.U.'s). Using the post encounter elements, apogee is calculated by

$$r_a = a(1 + e)$$

For this option we therefore must have

$$r_a \geq x \text{ A.U.}$$

Knowing the true anomaly at departure enables us to calculate* the time t_o from periapsis to that value of the anomaly. The true anomaly of the probe at $r = x$ A.U. is

$$\theta = \mathrm{Cos}^{-1}\left[\frac{a\left(1 - e^2\right) - r}{r\,e}\right]$$

and from this we may calculate the time $\left(t_1\right)$ from periapsis to x A.U. The functional value we then wish to minimize is

$$t_1 - t_o$$

Maximum Inclination to the ecliptic

From the position and velocity vectors ($\vec{R}$ and $\vec{V}$) of the post encounter trajectory we obtain the angular momentum vector $\vec{H} = \vec{R} \times \vec{V}$. If $\vec{k}^\circ$ is a unit vector perpendicular to the ecliptic then

$$\frac{\vec{H} \cdot \vec{k}^\circ}{|\vec{H}|} = \mathrm{Cos}\ i \text{ where } i \text{ is the inclination}$$

The maximum value of i occurs at 90° or $\mathrm{Cos}\ i = 0$. The functional value for the optimization process is $|\mathrm{Cos}\ i|$.

Maximum velocity component normal to the ecliptic

For this option we easily obtain that component of the velocity vector which is normal to the ecliptic from $\vec{V} \cdot \vec{k}^\circ$, where $\vec{V}$ and $\vec{k}^\circ$ are defined as above. In

*Reference 2, Subroutine TCONIC.

the ideal case this would occur at perigee and therefore the post encounter trajectory should have a true anomaly near zero.

Minimum perihelion distance

Perihelion (r_p) is determined from $r_p = a(1 - e)$. For this option it is desirable for the probe to be moving toward perigee so that perihelion is reached within a reasonable time. The optimization routine therefore constrains the true anomaly to be greater than 180 degrees.

Specific perihelion distance

In this option it is desired to determine a post encounter trajectory which makes r_p equal to a specified value. Again true anomaly is constrained to be greater than 180 degrees. The functional value to be minimized is the absolute value of the difference between the desired and actual values of r_p.

Additional Missions

The above options were analysed as examples of the class of mission objectives that are useful in a gravity assist program.

Further options that can be defined as functions of the post encounter trajectory parameters may be included and treated in a manner similar to the above missions.

IV. OPTIMIZATION PROCEDURE

As previously described, BMAG and ψ completely determine the post encounter trajectory for a given $\vec{V}_1$. This orbit is then used to obtain one of the post encounter missions described above.

However, for any given mission, the original independent variables will not be those which give the optimum for the mission desired. To accomplish the optimization of the particular mission, a multi-variable nonlinear iterator* is used which systematically alters the independent variables, calculates a new post encounter trajectory and a new functional value for the iterator to use in the next iteration. The functional value is that quantity to be maximized or minimized depending on the mission chosen. The iterating process continues until an optimum value is achieved.

*Reference 3.

V. EXAMPLES

In Tables 2 through 7, examples of each mission is given for five and six hundred day transfer times to the planet Jupiter. It is seen that some trajectories may be used for more than one purpose. For example the 500 day transfer for attaining a perihelion of .25 A.U. also may be used for out of the ecliptic studies since it reaches a distance of .9 A.U. from the ecliptic. As one might expect, missions out of the ecliptic show a certain symmetry in that the probe can pass above or below the planet with one trajectory being essentially an inversion of the other.

Tables 8 and 9 show that for a specific perihelion and for maximum inclination out of the ecliptic there is a range over which ψ may vary and still permit a successful mission to be performed. In Table 8, for ψ between $\pm 15°$ the perihelion distance of .25 A.U. is reached by varying BMAG. Table 9 shows that high inclination is attained along with useful distances out of the ecliptic for the ranges

$$10° < \psi \leq 60°$$

$$-60° \leq \psi < -10°$$

For $\psi = \pm 10°$, the probe will reach only about 1/2 A.U. out of the ecliptic before impacting the sun. However, for a multipurpose mission this may be acceptable.

Table 2

Time to 10 A.U. Jupiter Flyby

Flight Time (Days)	Time to 10 A.U. From Post Encounter (Days)	1/2 Time Spent in Sphere of Influence (Days)	Total Time to 10 A.U. (Days)	Miss Vector (km)	PSI (DEG)	RCA at Jupiter (km)
500	502.235	38.540436	1040.775436	1.1652525×10^6	178.59125	6.7396761×10^5
600	644.918	50.222080	1295.1401	1.5159822×10^6	178.29259	7.3006466×10^5

Table 3

Maximum Inclination to the Ecliptic

Flight Time (Days)	Max. Incl. to Ecliptic (DEG)	Miss Vector (km)	RCA at Jupiter (km)	Eccentricity	PSI (DEG)	Distances out of Ecliptic* (A.U.)	Period (Yrs.)
500	90.149125	9.2911205×10^5	4.7534129×10^5	.77527475	26.000001	-1.8041458 2.0249147	5.2807122
600	56.950182	1.8892977×10^6	1.0340760×10^6	.69967978	36.267950	-1.7966289 1.9774743	5.5981134
500	89.999997	9.8593254×10^5	5.2188549×10^5	.80545124	-27.277351	1.8033013 -1.7268608	5.1356543
600	54.728655	1.8857171×10^6	1.0310605×10^6	.68756694	-40.000001	1.8499558 -1.9160621	5.6568960

*In order of first one reached.
+above ecliptic
–below ecliptic

Table 4

Maximum Velocity Component Normal to the Ecliptic

Flight Time (Days)	Max. Vel. Comp. (km/sec.)	Distances out of Ecliptic (A.U.)	Inclination to Ecliptic (DEG)	True Anomaly (DEG)	Miss Vector (km)	RCA at Jupiter (km)	PSI (DEG)	Period (Yrs)
500	14.177346	hyperbolic	47.956324	1.2899815	6.5737490×10^5	2.6852945×10^5	89.882764	—
600	10.755579	-7.3077423 8.4287005	40.212817	0	1.1859282×10^6	4.8454809×10^5	89.360971	67.913349
500	13.699122	hyperbolic	47.607913	2.6708944	6.8282326×10^5	2.8652014×10^5	-90.265853	—
600	10.228707	6.4089444 -7.2540630	38.929086	1.8776355	1.2344030×10^6	5.1884239×10^5	-90.911229	51.194274

Table 5

Maximum Distance out of Ecliptic for 200 Days
After Exit From Sphere of Influence

Flight Time (Days)	Maximum Dist. Reached in 200 Days (A. U.)	Miss Vector (km)	PSI (DEG)	RCA at Jupiter (km)	Inclination to the Ecliptic (DEG)	Maximum Distances out of Ecliptic (A.U.)	Period (Yrs)
500	1.8583210	6.5752859×10^5	89.349005	2.6863708×10^5	48.230083	hyperbolic	—
600	1.4822333	1.1839131×10^5	89.280072	4.8313734×10^5	40.220488	-7.2762695 8.4449833	67.774794
500	1.9433977	6.8645633×10^5	-90.610745	2.8911606×10^5	47.489485	hyperbolic	—
600	1.5302251	1.2306858×10^6	-91.066911	5.1618858×10^5	38.839865	6.4452460 -7.3286843	52.469860

Table 6

Minimum Perihelion

Flight Time (Days)	Perihelion Distance (km)	Eccentricity	Semi-Major Axis (A. U.)	True Anomaly (DEG)	Period (Yrs)	Miss Vector (km)	RCA at Jupiter (km)	PSI (DEG)
500	2.3883212×10^{2}	.99999946	2.9372341	179.98286	5.0290366	1.6827966×10^{6}	1.1410724×10^{6}	-1.4842478
600	1.1801504×10^{7}	.97098112	2.7202921	-178.87158	4.4822933	2.1108910×10^{6}	1.2236820×10^{6}	-1.5013303

Table 7

Desired Perihelion Distance of .25 A.U.

Flight Time (Days)	Difference Between Desired and Actual Perihelion (km)	Eccentricity	Inclination to the Ecliptic (DEG)	True Anomaly (DEG)	Period (Yrs)	Miss Vector (km)	RCA at Jupiter (km)	PSI (DEG)
500	2.5	.91343500	99.575354	-174.88444	4.9031313	8.0956680×10^5	3.8072240×10^5	14.654366
600	-7.5	.92040639	13.77936	-170.17053	5.5612133	1.1967380×10^6	4.9213574×10^5	-6.8018565

Table 8

.25 A.U. Perihelion Range of Convergence

Flight Time (Days)	Difference Between Actual and Desired Perihelion	Eccentricity	Inclination to the Ecliptic (DEG)	True Anomaly (DEG)	Period (Yrs)	Miss Vector (km)	RCA at Jupiter (km)	PSI (DEG)
500	-2.0	.93989170	41.982970	-164.45478	8.4739364	4.4370779×10^5	1.3353651×10^5	10.0
500	2.5	.91343500	99.575354	-174.88444	4.9031313	8.0956680×10^5	3.8072240×10^5	14.654366
500	-5.0	.93503752	53.120298	-165.71581	7.5421111	4.8431230×10^5	1.5664095×10^5	-15.0
500	3.61×10^7	.86569943	70.820855	-162.00093	6.9917864	5.4603428×10^5	1.9423521×10^5	20.0
500	2.29×10^7	.89522925	60.934946	-162.10933	7.5506916	5.0109828×10^5	1.6658172×10^5	-20.0
500	0	.94683354	5.3318050	-162.83654	10.186624	3.9075805×10^5	1.0556918×10^5	0

Table 9

Maximum Inclination

Range of Convergence

Flight Time (Days)	Max Inclination to Ecliptic	Miss Vector (km)	RCA at Jupiter (km)	Eccentricity	PSI (DEG)	Distances Out Ecliptic	Period (Yrs)
500	67.429855	1.0257742×10^6	5.5503884×10^5	.20531143	60.0	-5.0332479 3.3110636	11.522529
500	67.038053	1.0264920×10^6	5.5563983×10^5	.23098417	-60.0	4.4198905 -2.9545003	10.377564
500	83.355080	1.0282312×10^6	5.5709629×10^5	.64723460	35.0	-2.5991028 2.3602517	5.9349354
500	83.444641	1.0283467×10^6	5.5719315×10^5	.69554280	-35.0	2.4177092 -2.1151134	5.6721791
500	90.149125	9.2911205×10^5	4.7534129×10^5	.77527475	26.000001	-1.8041458 2.0249147	5.2807122
500	89.999997	9.8593254×10^5	5.2188549×10^5	.80545124	-27.277351	1.8033013 -1.7268608	5.1356543
500	89.857761	6.5224351×10^5	2.6494382×10^5	.9589746	10.0	-.53530149 1.2141945	5.3760529
500	89.987443	6.5006698×10^5	2.6342724×10^5	.97634397	-10.0	.47548276 -.84372660	5.3138196

VI. REFERENCES

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2. Programmer's Manual for the Quick Look Mission Analysis Program, WDL-TR2217 Philco Corporation, Palo Alto, California, January 24, 1964.

3. Ferguson, R.; Smith, D.; Orlow, T., 7090 Computer Program Minmax-A Generalized Nonlinear Iterator, TN-7002 (M-67), U.S. Naval Ordnance Laboratory, White Oak, Maryland, October 18, 1965.

APPENDIX A

Maximum Distances out of the Ecliptic

When the post encounter trajectory takes the probe out of the ecliptic plane it would be meaningful for certain missions to know the maximum distance out of the ecliptic that the probe attains. Clearly this implies that the trajectory is an ellipse. Such a trajectory with a relatively small period would be extremely useful in studying those regions of space above and below the ecliptic.

Figure 6 shows a typical post encounter trajectory which is inclined to the ecliptic by an angle i. The vector $\vec{e}$ is directed toward perigee; $\vec{H} = \vec{R} \times \vec{V}$ and is perpendicular to the orbital plane and $\vec{k}°$ is a unit vector perpendicular

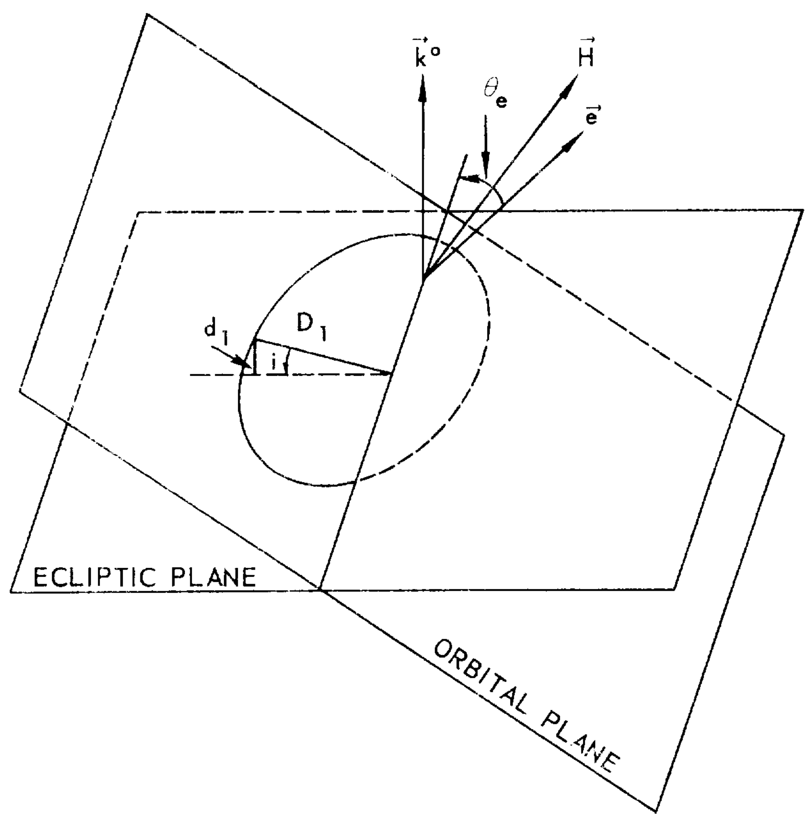

Figure 6

to the ecliptic. θ_e is the angle formed by the vector $\vec{e}$ and the vector $\vec{k}^\circ \times \vec{H}$ which lies along the line of intersection of the two planes. The angles i and θ_e are easily found from the post encounter trajectory as follows:

$$\text{Cos } i = \frac{\vec{H} \cdot \vec{k}^\circ}{|\vec{H}|}$$

$$\text{Cos } \theta_e = \frac{(\vec{k}^\circ \times \vec{H}) \cdot \vec{e}}{|\vec{k}^\circ \times \vec{H}| \; |\vec{e}|}$$

To compute the maximum distances both above and below the ecliptic it is convenient to consider a coordinate system with origin at the center of the ellipse and with the X axis along $\vec{e}$. Referring to Figure 7 we compute the maximum distance from the ecliptic at the point (X_o, Y_o).

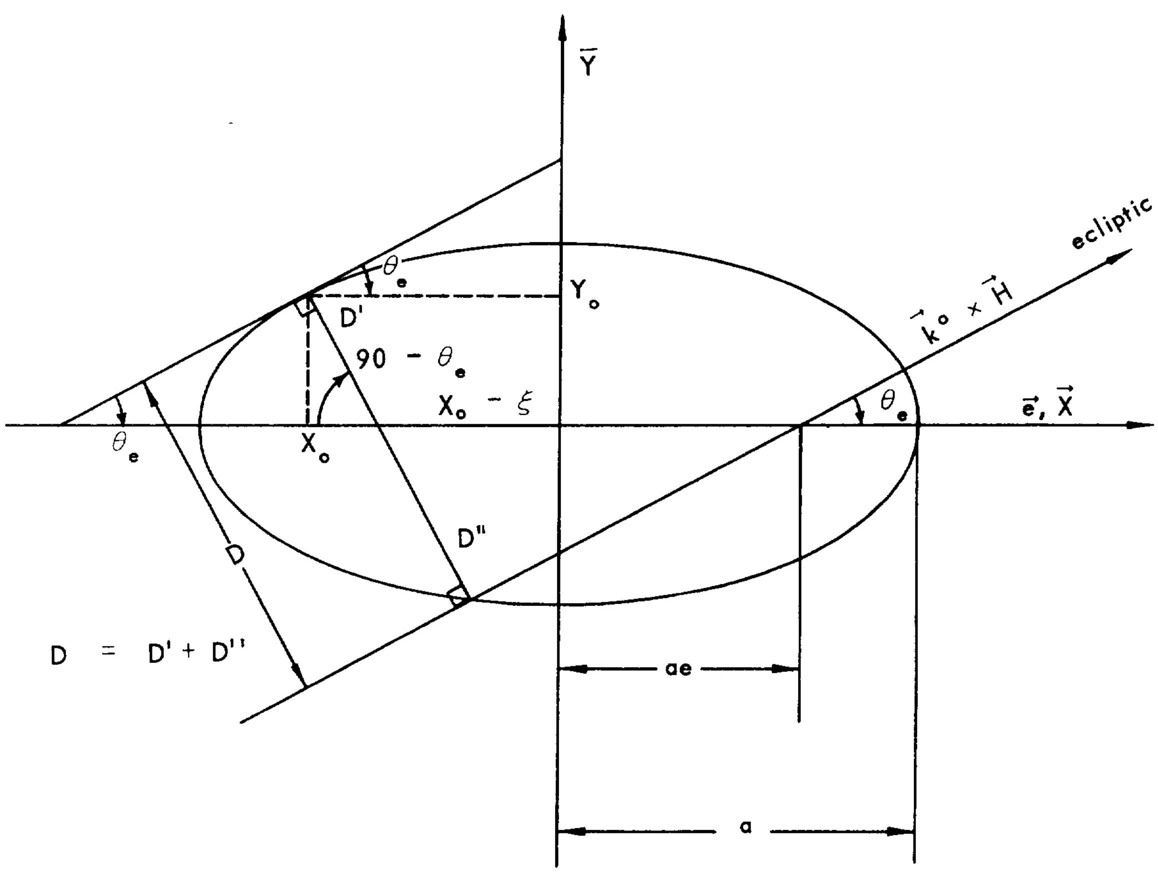

Figure 7

From the post encounter trajectory, a and $e = |\vec{e}|$ are already known. The equation of an ellipse is

$$y = \pm b \left(1 - \frac{X^2}{a^2}\right)^{\frac{1}{2}} \qquad \text{where}$$

$$b = a\sqrt{1 - e^2}$$

and

$$\frac{dy}{dx} = \pm \frac{Xb}{a^2}\left(1 - \frac{X^2}{a^2}\right)^{-\frac{1}{2}}$$

at (X_o, Y_o) we must have

$$X_o \frac{b}{a^2}\left(1 - \frac{X_o^2}{a^2}\right)^{-\frac{1}{2}} = \pm \operatorname{Tan} \theta_e$$

so that

$$X_o{}^2 = \frac{a^4 \operatorname{Tan}^2 \theta_e}{b^2 + a^2 \operatorname{Tan}^2 \theta_e} \qquad \text{substituting for b}$$

$$X_o{}^2 = \frac{a^2 \operatorname{Tan}^2 \theta_e}{1 - e^2 + \operatorname{Tan}^2 \theta_e}$$

$$\text{then} \quad Y_o = a\left[\left(1 - e^2\right)\left(1 - \frac{X_o{}^2}{a^2}\right)\right]^{\frac{1}{2}}$$

where we consider only the absolute values of X_o and Y_o. From Figure 7

$$D' = \frac{Y_o}{\operatorname{Cos} \theta_e}$$

$$\operatorname{Sin} \theta_e = \frac{D''}{X_o - \xi + ae}; \quad X_o \geq 0$$

thus $D'' = X_o \operatorname{Sin} \theta_e - \xi \operatorname{Sin} \theta_e + ae \operatorname{Sin} \theta_e$

also $\operatorname{Tan} 90 - \theta_e = \cot \theta_e = \frac{Y_o}{\xi} \; ; \; Y_o \geq 0$

then $\xi = \frac{Y_o \operatorname{Sin} \theta_e}{\operatorname{Cos} \theta_e}$

finally $D = D' + D'' = X_o \operatorname{Sin} \theta_e - \frac{Y_o \operatorname{Sin}^2 \theta_e}{\operatorname{Cos} \theta_e} + ae \operatorname{Sin} \theta_e + \frac{Y_o}{\operatorname{Cos} \theta_e}$

$$\text{or} \quad D = X_o \operatorname{Sin} \theta_e + Y_o \operatorname{Cos} \theta_e + ae \operatorname{Sin} \theta_e \tag{1}$$

for the distance on the opposite side of the ecliptic we obtain similarly

$$D_2 = Y_o \operatorname{Cos} \theta_e + X_o \operatorname{Sin} \theta_e - ae \operatorname{Sin} \theta_e \tag{2}$$

Finally, the distances out of the ecliptic are

$$d_1 = D \operatorname{Sin} i$$

$$d_2 = D_2 \operatorname{Sin} i \, .$$

The distance computed using equation (1) will be the maximum distance (d_1) attained by the probe. That distance obtained from equation (2) is the maximum distance on the other side of the orbit (d_2). If $\vec{k}^\circ \cdot \vec{e} > 0$ then d_1 is positive and above the ecliptic while d_2 is below the ecliptic; the opposite is true if $\vec{k}^\circ \cdot \vec{e}$ is less than zero.

APPENDIX B

BLOCK DIAGRAM OF COMPUTER PROGRAM